ON THE CONDITIONS AT ELASTIC WAVE FRONTS PROPAGATING IN A NONHOMOGENEOUS MEDIUM

(OB USLOVIIAKH NA FRONTAKH UPRUGIKH VOLN RASPOSTRANIAIUSHCHIKHSIA V NEODNORODNOI SREDE)

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Two types of waves can be propagated independently from one another in a homogeneous elastic medium. In the longitudinal waves the displacements $\mathbf{u}$ are such that there are no element rotations (rot $\mathbf{u} = 0$), and in transverse waves there are no volume changes (div $\mathbf{u} = 0$).

In a non-homogeneous elastic medium no independent longitudinal and transverse waves exist. Volume changes as well as element rotations are simultaneously present in these wave motions.

This article deals with the study of the character of the displacements near the wave front (a surface at which the displacements undergo a finite discontinuity), which moves in an ideally elastic nonhomogeneous medium. Fronts with such discontinuities may correspond to sources of waves with a time dependence of the type of a step force (Heaviside functions), or an integral of it [1].

Let some finite volume V be isolated in the medium and assume that the properties of the material inside V are such that the elastic Lame parameters λ and μ vary continuously, together with their derivatives, and the density ρ is a continuous function of the coordinates.

The displacements in wave motions depend on the coordinates and the time, and can thus be studied as vector point functions in a four-dimensional space-time.

The wave front traverses the entire volume V in a finite time. Thus the entire process can be assumed to take place in a bounded region of the space-time G.

Corresponding to the moving front in region G there is a hypersurface Γ, on which the displacements undergo a finite jump. Assume that there exists only one surface of discontinuity Γ which divides G into two parts

G_1 and G_2.

Let us introduce a Cartesian coordinate system x_1, x_2, x_3. The equations of motion in this coordinate system can be written in the following form:

$$\sum_{j=1}^{3} \frac{\partial \tau_{ij}}{\partial x_j} - \rho \frac{\partial^2 u_i}{\partial t^2} = 0 \tag{1}$$

where τ_{ij} are stress components and u_i are displacement components.

According to our assumption the nonhomogeneous medium obeys Hooke's law, i.e.

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda \delta_{ij} \sum_{n=1}^{3} \frac{\partial u_n}{\partial x_n} \tag{2}$$

Here δ_{ij} is the Kronecker delta. The operator on the vector function $\mathbf{u}(x, t)$ is obtained by substitution of (2) into (1) and denoted by L. A function that has a finite discontinuity inside the region cannot be the ordinary solution of the system of differential equations (1). Thus we require that the discontinuous function be the generalized solution in the Sobolev [2] sense.

The vector function $\mathbf{u}(x, t)$ is called the generalized solution of the equations of the theory of elasticity in the region G, if for any vector function $\mathbf{f}(x, t)$, which is twice continuously differentiable and which goes to zero together with its first derivatives on the boundary, the following holds good:

$$\int_G (\mathbf{u} \cdot L\mathbf{f})\, d^3x\, dt = 0 \tag{3}$$

Let us introduce the quantities

$$\tau_{ij}^* = \mu \left(\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i}\right) + \lambda \delta_{ij} \sum_{n=1}^{3} \frac{\partial f_n}{\partial x_n} \tag{4}$$

which are related to the components of the vector function $\mathbf{f}$ as the stress components are related to the displacements. All quantities τ_{ij}^* go to zero on the boundary of the region G.

The operator L is self-adjoint, and the expression

$$(\mathbf{f} \cdot L\mathbf{u}) - (\mathbf{u} \cdot L\mathbf{f}) = \sum_{i=1}^{3} \frac{\partial P_i}{\partial x_i} + \frac{\partial P_t}{\partial t} \tag{5}$$

has the appearance of a four-dimensional divergence. In this expression

$$P_i = \sum_{j=1}^{3} (f_j \tau_{ij} - u_j \tau_{ij}^*) \tag{6}$$

$$P_t = -\rho \sum_{j=1}^{3} \left(f_j \frac{\partial u_j}{\partial t} - u_j \frac{\partial f_j}{\partial t} \right) \tag{7}$$

Let us assume that the equation of the discontinuity hypersurface Γ can be written in the form $t - \psi(x_1, x_2, x_3) = 0$.

Let us split the integral (3) into integrals over G_1 and G_2, substitute the expressions from (5) and transform the integrals containing the four-dimensional divergence to surface integrals by the Gauss-Ostrogradskii formula. We obtain

$$\int_{G_1} (\mathbf{f} \cdot L\mathbf{u})\, d^3x\, dt + \int_{G_2} (\mathbf{f} \cdot L\mathbf{u})\, d^3x\, dt - \tag{8}$$
$$- \int_{\Gamma} \left\{ \sum_{i=1}^{3} P_i \frac{\partial \psi}{\partial x_i} - P_t \right\} \frac{dS}{\sqrt{(\nabla\psi)^2 + 1}} + \int_{\Gamma} \left\{ \sum_{i=1}^{3} P_i \frac{\partial \psi}{\partial x_i} - P_t \right\} \frac{dS}{\sqrt{(\nabla\psi)^2 + 1}} = 0$$

The surface integrals extend over different sides of the surface Γ. Combining them into one integral we obtain

$$\int_{\Gamma} \left\{ \sum_{i=1}^{3} [P_i] \frac{\partial \psi}{\partial x_i} - [P_t] \right\} \frac{dS}{\sqrt{(\nabla\psi)^2 + 1}},$$

where $[P_i]$ and $[P_t]$ are differences of the corresponding values (jumps) of the quantities on different sides of Γ. In the subsequent discussion the square brackets will be omitted, with the implication that there are discontinuities in the functions wherever such occur. We transform the expressions under the surface integral so that they contain only the functions f_i, their derivatives with respect to the directions lying in the plane tangent to Γ, and derivatives with respect to the normal to the surface Γ. The terms containing the derivatives in the tangential directions are of the form

$$\int_{\Gamma} \Phi \frac{\partial f_i}{ds}\, dS = - \int_{\Gamma} f_i \frac{\partial \Phi}{\partial s}\, dS \tag{9}$$

where Φ is some function, and the differentiation takes place in a direction tangent to Γ. Here we used integration by parts, and the fact that f_i goes to zero on the boundaries of region G. Formula (8) can be rewritten in the form

$$\int_{G_1} (\mathbf{f} \cdot L\mathbf{u})\, d^3x\, dt + \int_{G_2} (\mathbf{f} \cdot L\mathbf{u})\, d^3x\, dt - \int_{\Gamma} \left\{ (\mathbf{a} \cdot \mathbf{f}) + \left(\mathbf{b} \cdot \frac{\partial \mathbf{f}}{\partial \nu} \right) \right\} \frac{dS}{\sqrt{\nabla\psi)^2 + 1}} = 0 \tag{10}$$

where

$$\mathbf{a} = 2\mu\, (\nabla\psi \cdot \nabla)\, \mathbf{u} + (\lambda + \mu)\, \nabla\, (\nabla\psi \cdot \mathbf{u}) + (\lambda + \mu)\, \nabla\psi\, \mathrm{div}\, \mathbf{u} + \mu \triangle \psi\, \mathbf{u} + (\nabla\mu \cdot \nabla\psi)\, \mathbf{u} +$$
$$+ (\nabla\mu\mathbf{u})\, \nabla\psi + (\nabla\psi, \mathbf{u})\, \nabla\lambda + \mu\, (\nabla\psi)^2 \frac{\partial \mathbf{u}}{\partial t} + (\lambda + \mu) \left(\nabla\psi, \frac{\partial \mathbf{u}}{\partial t} \right) \nabla\psi + \rho \frac{\partial \mathbf{u}}{\partial t} \tag{11}$$

$$\mathbf{b} = (\lambda + \mu)(\nabla\psi, \mathbf{u})\, \nabla\psi - \{\rho - \mu\, (\nabla\psi)^2\}\, \mathbf{u} \tag{12}$$

It follows from the fact that (10) has to be satisfied by any vector function with the properties shown above, that $L\mathbf{u} = 0$ in the regions G_1 and G_2, and the conditions $\mathbf{a} = 0$ and $\mathbf{b} = 0$ are satisfied on the surface Γ. The generalized solution should be the ordinary solution in the regions where it is continuous together with its derivatives, and it should satisfy the additional conditions on the discontinuity surface.

The equation $\mathbf{b} = 0$ was analyzed in reference [3]. It is a system of linear homogeneous equations in terms of the components of $\mathbf{u}$. When the determinant of the system is equated to zero, for the function ψ characterizing the wave front, i.e. the surface of discontinuity, we obtain the following equation:

$$\{\rho - \mu (\nabla\psi)^2\}^2 \{\rho - (\lambda + 2\mu)(\nabla\psi)^2\} = 0 \tag{13}$$

The first bracket fives the equation for a front traveling at the speed of transverse waves, and the second that for a front traveling at the speed of longitudinal waves. As has been shown in [3], the displacement jump in the transverse wave is necessarily perpendicular to the normal of the wave front, and in the longitudinal wave is necessarily parallel to the normal of the front. Using these properties and equation (13), further results can be obtained from the equation $\mathbf{a} = 0$.

First let us study the condition $\mathbf{a} = 0$ at the front of a longitudinal wave. If this equation is multiplied scalarly by the unit vector normal to the front $\mathbf{t}$, the differential equation derived in [4] and [5], describing the change of the intensity of the discontinuity along a ray, can be obtained. If the condition $\mathbf{a} = 0$ is multiplied vectorially by $\mathbf{t}$ and some simple transformations are performed, a formula for the discontinuity of rot $\mathbf{u}$ at the front of the longitudinal wave can be obtained:

$$\operatorname{rot} \mathbf{u} = \frac{\mathbf{u} \times \mathbf{P}}{\lambda + \mu} \qquad (\mathbf{P} = a^2\nabla\rho - 2\nabla\mu) \tag{14}$$

where $\mathbf{u}$ is the discontinuity of the displacement vector at the front, and a is the speed of propagation of the longitudinal wave.

Similarly, the condition $\mathbf{a} = 0$ at the front of a transverse wave can be studied. When this condition is multiplied vectorially by $\mathbf{t}$, a differential equation for the change of the intensity of the discontinuity at the front of a transverse wave along a ray [4], [5] can be obtained. When this condition is multiplied scalarly by t and some simple transformations are performed, then a formula for the discontinuity of the divergence at the front of a transverse wave can be reached:

$$\operatorname{div} \mathbf{u} = \frac{(\mathbf{u} \cdot \mathbf{S})}{\lambda + \mu} \qquad (\mathbf{S} = b^2\nabla\rho - 2\nabla\mu) \tag{15}$$

where $\mathbf{u}$ is the discontinuity of the displacement vector at the front of the transverse wave, and b is the speed of propagation of transverse waves.

Knowing the displacement discontinuities at the front, from (14) and (15) we can compute the discontinuities of the rotation or divergence of displacements corresponding to the fronts of longitudinal and transverse waves. If the initial conditions for the displacement jumps are known, then they can be computed for all points through which the front passes by solving the ordinary differential equations derived in [4], and [5].

It can be seen that in the homogeneous medium the discontinuities of the rotation of the displacements at the front of a longitudinal wave and the divergence of the displacements at the front of a transverse wave are equal to zero.

Formulas (14) and (15) also make sense for continuously varying displacement fields. If at every point of the space the displacements change according to the law $f(t)$ after the time t_0 at which the front passed that point, then the error in using formulas (14) and (15) would be of the order

$$f_1(t) = \int_{t_0}^{t} f(\tau)\, d\tau$$

Such an estimate can be obtained by constructing a continuous solution out of the discontinuous ones by means of the Duhamel integral.

For a rapidly varying function $f(t)$, the errors near the wave front can be neglected, and it can be assumed that in this zone the formulas obtained relate the values of the displacements **u** to the values of div **u** and rot **u** respectively. This result may be obtained by a method of expanding the solution near the front as described in reference [4].

BIBLIOGRAPHY

1. Ogurtsov, E.I., Uspenskii, I.N. and Ermilova, N.I., Nekotorye kolichestvennye issledovaniia po rasprostraneniiu voln v prosteishikh uprugikh sredakh, Sb. "*Voprosy dinamicheskoi teorii rasprostraneniia seismicheskikh voln*" (Some quantitative studies of wave propagation in the simplest elastic media, Coll. "*Problems of the dynamical theory of propagation of seismic waves*"). Gostoptekhizdat, No. 1, 1957.

2. Sobolev, S.L., *Nekotorye primeneniia funktsional'nogo analiza (Some applications of functional analysis)*. Izd. LGU, 1950.

3. Levin, M.L. and Rytov, S.M., O perekhode k geometricheskomu priblizheniiu v teorii uprugosti (On the transition to the geometrical approximation in the theory of elasticity). *Akust. Zh.* No. 2, 1956.

4. Babich, V.M. and Alekseev, A.S., O luchevom metode vychisleniia intensivnosti volnovykh frontov (On the ray method of computing the intensity of wave fronts). *Izv. AN SSSR, ser. Geofiz.* No. 1, 1958.

5. Skuridin, G.A. and Gvozdev, A.A., O kraevykh usloviiakh dlia skachkov razryvnykh reshenii dinamicheskikh uravnenii teorii uprugosti (On boundary conditions for the jumps of discontinuous solutions of the dynamical equations of the theory of elasticity). *Izv. AN SSSR, ser. Geofiz.* No. 2, 1958.

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